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Helical flows of fractional viscoelastic fluid in a circular pipe

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ABSTRACT

The exploration of this study is devoted to investigate the helical effects for the flow of fractionalized viscoelastic fluid in helically moved cylinder. The cylinder starts to oscillate and rotate about its axis when $t = 0^+$ with velocities. By applying mathematical transforms (Hankel and discrete Laplace transforms) exact solutions are found out for velocities and shear stresses. The general solutions satisfy initial conditions $u_1(r,0) = u_2(r,0) = \frac{\partial u_1(r,0)}{\partial t} = \frac{\partial u_2(r,0)}{\partial t} = 0$, as well as boundary conditions $u_1(R,t) = R\Omega H(t)\,sin(\omega t)$ or $cos(\omega t)$, and $u_2(R,t) = UH(t)sin\omega t/cos\omega t$. The solutions are presented in terms of series form and expressed in terms of generalized Fox H-function $H^{1,j}_{j,k+1}(Z)$. Special cases have been traced out for non-Newtonian fluids (fractional and ordinary Second Grade, fractional and ordinary Newtonian fluid and ordinary Maxwell Fluid). Three types of fluid models are presented for rheological comparison, namely (i) fractional and ordinary Maxwell fluid, (ii) fractional and ordinary second grade fluid and (iii) fractional and ordinary Newtonian fluid. Finally, the rheology is influenced with distinct parameters and material limitations for helically moved cylinder by depicting graphical analysis.

1. Introduction

The practical applications of flow of non-Newtonian fluids lie among the modern industries. Such industries have diverted the attention of engineers, mathematicians and scientists for the solutions of flow problems of non-Newtonian fluids. Various rheological materials like polymer melts, suspensions, clay coatings, drilling mud, elastomers, certain greases and oils, and numerous emulsions are considered as non-Newtonian fluids. In order to exhibit certain characteristics of non-Newtonian fluids there is not at least single constitutive equation, this is due to complex behavior of fluid. The rheology of non-Newtonian fluids has distinct characteristics in fermentation, boiling, polymer processing, molten plastic foam processing, composite processing many others (Fetecau and Corina, 2005; Fetecau et al., 2007; Erdogan and Imrak, 2005; Chen et al., 2004; Hayat et al., 2004a; 2004b; Abro and Shaikh, 2015). Nowadays, viscoelastic fluid (Maxwell model) is acknowledged by many scientists and engineers in several engineering and industrial processes. This is due to involvement of viscoelastic material, for instance, glues, paints, melts of polymers, asphalts, biological solutions, colloids and several others. In continuation, Hayat et al. (2006) investigated flow problem for second grade fluid in cylindrical geometry in which they traced out analytical solutions. Analytical solutions have been obtained by Akl (2014) on the structure of stretching cylinder for unsteady boundary flow. Altintas and Ozkol (2015) analyzed non-heated and heated cases in circular pipes for magnetohydrodynamic flow. Sulochana and Sandeep (2016) worked at different temperature for shrinking cylinder with heat transfer behavior of magnetohydrodynamics. Masood et al. (2016) investigated a stagnation-point flow with the nonlinear radiative on Sisko fluid over stretching cylinder. They established numerical solutions via shooting method through forth order Runge-Kutta method by transforming governing partial differential equation of stretching cylinder. They also explored nonlinear Rosseland approximation and effects of thermal radiation. Jamil et al. (2011) analyzed longitudinal and torsional constantly an infinite accelerated cylinder with for second order liquid. They found exact analytical solutions for shear stress and velocity profile. Mahmood et al. (2010) worked on the annular region of cylinders for generalized second order liquid with oscillatory flow. They utilized integral transform to investigate some exact analytical solutions with few limiting

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cases. Siddique and Vieru (2009) examined circular cylinder for rotational fluid of second order liquid and investigated analytical study in cylindrical configuration. Abdulhameed et al. (2016) perceived oscillating flow in circular cylinder for heat performance and compared due to different pressure waveforms. Sulochana and Sandeep (2016) studied heat transfer and momentum behavior of few nanoparticles embedded towards porous cylinder. They obtained numerical solutions by employing Runge-Kutta Felhberg technique. Of course the list of study on circular cylinder for viscoelastic fluid can be continuous but we end it with some recent references (Shah and Qi, 2010; Wang and Xu, 2009; Nazar et al., 2010; Fetecau et al., 2010; Malekzadeh et al., 2011; Abro and Solangi, 2017; Muhammad et al., 2015; Abro, 2016; Rostami et al., 2014; Rashidi et al., 2014; Rashidi et al., 2012; Rashidi et al., 2015). Motivating by above studies, our purpose is to investigate the helical effects for the flow of fractionalized viscoelastic fluid in helically moved cylinder. The cylinder starts to oscillate and rotate about its axis when $t=0^+$ with velocities. By applying mathematical transforms (Hankel and discrete Laplace transforms) exact solutions are found out for velocities and shear stresses. The solutions are presented in terms of series form and expressed in terms of generalized Fox H-function $H_{j,k+1}^{1,j}(Z)$. Special cases have been traced out for non-Newtonian fluids (fractional and ordinary Second Grade, fractional and ordinary Newtonian fluid and ordinary Maxwell Fluid). Three types of fluid models are presented for rheological comparison, namely (i) fractional and ordinary Maxwell fluid, (ii) fractional and ordinary second grade fluid and (iii) fractional and ordinary Newtonian fluid.. Finally, the rheology is influenced with distinct parameters and material limitations among two helically moved cylinders by depicting graphical analysis.

2. Mathematical modeling of helices

The Cauchy stress tensor T is an incompressible Maxwell fluid is given (Fetecau et al., 2010; Abro, 2016)

$$\mathrm{T}=-p\mathrm{I}+\mathrm{S},\mathrm{S}+\Lambda\left(\dot{\mathrm{S}}-\mathrm{LS}-\mathrm{SL}^{\mathrm{T}}\right)=\mu\mathrm{A}, \tag{1}$$

where $-p\mathrm{I}, \mathrm{S}, \mathrm{L}, \mathrm{A}=\mathrm{L}+\mathrm{L}^{\mathrm{T}}, \mu, \Lambda, T$ are the indeterminate spherical stress due to the constraint of incompressibility, the extra-stress tensor, velocity gradient, first Rivilvin Ericksen tensor, dynamic viscosity, relaxation time, transpose operation. This Maxwell model can also be characterized for Newtonian fluid by letting $\Lambda\to 0$. The microscopic polymers and their predictions of the normal-stress differences are also characterized by this model. Due to this significance, this model is useful to analyze dilute polymeric fluids in viscoelasticity. Here velocity field is assumed as

$$u=u(r,t)=u_1(r,t)e_\theta+u_2(r,t)e_z,\quad \mathrm{S}=\mathrm{S}(r,t), \tag{2}$$

where, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_z$ are unit vectors in the θ and z-direction. For such flows the constraint of incompressibility is automatically satisfied. If the fluid is at rest up to the moment $t=0$ then

$$u(r,0)=\mathrm{S}(r,0)=0, \tag{3}$$

and implementing $S_{rr}=0$ in Eq. 1 then we arrived at meaning full equations as defined below

$$\Lambda\frac{\partial}{\partial t}\tau_1(r,t)-\mu\frac{\partial u_2(r,t)}{\partial r}+\mu\frac{u_2(r,t)}{r}+\tau_1(r,t)=0, \tag{4}$$
$$\Lambda\frac{\partial}{\partial t}\tau_2(r,t)-\mu\frac{\partial u_2(r,t)}{\partial t}+\tau_2(r,t)=0, \tag{5}$$

here, $\tau_1=S_{r\theta}$ and $\tau_2=S_{rz}$ are the shear stresses. While, due to nonappearance of pressure gradient, balance of linear momentum and ignoring body the forces lead the following equation for symmetry of rotation as

$$\rho\frac{\partial u_1(r,t)}{\partial t}-\frac{\partial\tau_1(r,t)}{\partial r}-\frac{2\tau_1(r,t)}{r}=0, \tag{6}$$
$$\rho\frac{\partial u_2(r,t)}{\partial t}-\frac{\partial\tau_2(r,t)}{\partial r}-\frac{\tau_2(r,t)}{r}=0, \tag{7}$$

eliminating τ_1 and τ_2 from Eqs. 4-7, we arrive at the equations governs the helical flow Fetecau et al. (2008)

$$\frac{\partial u_1(r,t)}{\partial t}+\Lambda\frac{\partial}{\partial t}\left(\frac{\partial u_1(r,t)}{\partial t}\right)-\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)u_1(r,t)=0,\quad t>0, \tag{8}$$
$$\frac{\partial u_2(r,t)}{\partial t}+\Lambda\frac{\partial}{\partial t}\left(\frac{\partial u_2(r,t)}{\partial t}\right)-\nu\left(\frac{\partial^2}{\partial r^2}+\frac{\nu}{r}\frac{\partial}{\partial r}\right)u_2(r,t)=0,\quad t>0, \tag{9}$$

where $\nu=\frac{\mu}{\rho}$ is the kinematic viscosity of the fluid. Meanwhile, we consider here fractional Maxwell fluid at rest in an oscillating circular cylinder of radius R. At time $t=0^+$ the cylinder begins to rotate about its own axis ($t=0$) with the angular velocity $\Omega\sin(\omega t)$ or $\Omega\cos(\omega t)$ and oscillates along the same axis with $U\sin(\omega t)$ or $U\cos(\omega t)$. Due to shear the fluid is gradually moved and its velocity being of the pattern as in (2), while the governing equations are (4-5) and (8-9). Such flow produces helicity, this is due to fact that the streamline of helicity are helices as shown in Fig. 1. The conditions are:

- Initial condition:

$$u_1(r,0)=u_2(r,0)=\tau_1(r,0)=\tau_2(r,0)=\frac{\partial u_1(r,0)}{\partial t}=\frac{\partial u_2(r,0)}{\partial t}=0, \tag{10}$$

- Boundary condition for angular velocity:

$$\begin{aligned}&u_1(R,t)=R\Omega H(t)\sin(\omega t)\ or\ R\Omega H(t)\cos(\omega t),\ t\geq 0\\ &u_2(R,t)=UH(t)\sin(\omega t)\ or\ UH(t)\cos(\omega t),\quad t\geq 0,\end{aligned} \tag{11}$$

where $H(t)$ is Heaviside function. In order to develop the governing equations for helical flow (4-5) and (8-9) in terms of non-integer order derivative, we implement the Caputo-fractional operator, we get

$$\frac{\partial u_1(r,t)}{\partial t}+\Lambda D_t^\chi\left(\frac{\partial u_1(r,t)}{\partial t}\right)-\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)u_1(r,t)=0,\quad t>0, \tag{12}$$

$$\frac{\partial u_2(r,t)}{\partial t} + \Lambda D_t^{\chi}\left(\frac{\partial u_2(r,t)}{\partial t}\right) - \nu\left(\frac{\partial^2}{\partial r^2} + \frac{\nu}{r}\frac{\partial}{\partial r}\right)u_2(r,t) = 0, \quad t > 0, \tag{13}$$

$$\Lambda D_t^{\chi}\tau_1(r,t) - \mu\frac{\partial u_1(r,t)}{\partial r} + \mu\frac{u_1(r,t)}{r} + \tau_1(r,t) = 0, \tag{14}$$

$$\Lambda D_t^{\chi}\tau_2(r,t) - \mu\frac{\partial u_2(r,t)}{\partial t} + \tau_2(r,t) = 0, \tag{15}$$

where, D_t^{χ} represent the non-integer order Caputo fractional operator defined as (Abro et al., 2016; Abro et al., 2017a)

$$D_t^{\chi}u(t) = \begin{cases} \frac{1}{\Gamma(1-\chi)}\int_0^t \frac{u'(q)}{(t-q)^{\chi}}dq\,, & 0 < \chi < 1; \\ \frac{du(t)}{dt}\,, & \chi = 1 \end{cases}, \tag{16}$$

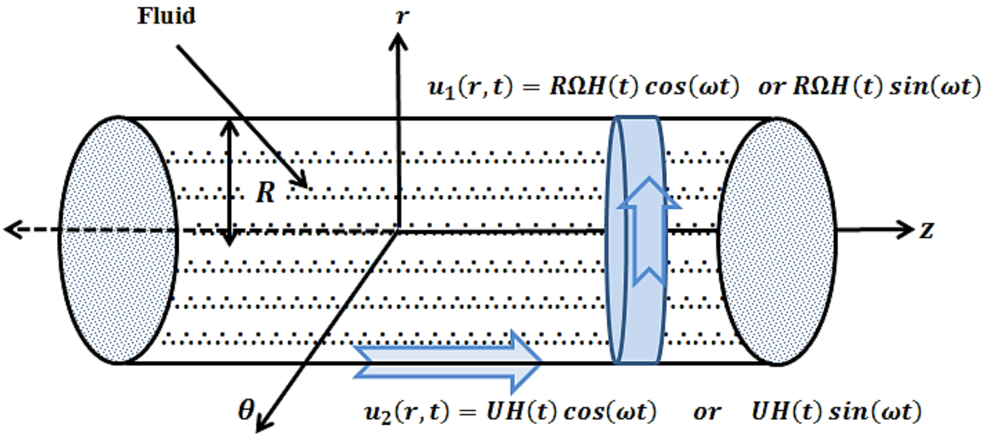

Fig. 1: Geometrical configuration of helecity

3. Solution of the problem

3.1. Velocity field

- **Case-I: For sine oscillations:** Applying Laplace transform on Eqs. 12-13 and keeping in mind Eqs. 10-11, we arrive at

$$(s + \Lambda^{\chi}s^{1+\chi})\bar{u}_1(r,s) = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\bar{u}_1(r,s), \tag{17}$$

$$(s + \Lambda^{\chi}s^{1+\chi})\bar{u}_2(r,s) = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\bar{u}_2(r,s), \tag{18}$$

where the conditions $\bar{u}_1(r,s) = \frac{R\Omega\omega}{(s^2+\omega^2)}$ and $\bar{u}_2(r,s) = \frac{U\omega}{(s^2+\omega^2)}$ are to satisfy Eqs. 17-18. Employing finite Hankel transform on Eqs. 17-18 and using Appendix A (A1 and A3) on Eq. 17 and (A2, A4) on Eq. 18, we get

$$\bar{u}_{1H}(r_\alpha, s) = \frac{\Omega\, J_2(Rr_\alpha)\,\omega\,\nu\,R^2 r_\alpha}{s^2(\Lambda^{\chi}s^{1+\chi}+s+\nu r_\alpha^2)+\omega^2(\Lambda^{\chi}s^{1+\chi}+s+\nu r_\alpha^2)}, \tag{19}$$

$$\bar{u}_{2H}(r_\beta, s) = \frac{U\, J_1(Rr_\beta)\,\omega\,\nu\,R\,r_\beta}{s^2(\Lambda^{\chi}s^{1+\chi}+s+\nu r_\beta^2)+\omega^2(\Lambda^{\chi}s^{1+\chi}+s+\nu r_\beta^2)}, \tag{20}$$

in order to satisfy imposed conditions, we present suitable equivalent forms of Eqs. 19-20 as

$$\bar{u}_{1H}(r_\alpha, s) = \frac{\Omega\, J_2(Rr_\alpha)\,\omega\,R^2}{r_\alpha(s^2+\omega^2)} - \frac{\Omega\, J_2(Rr_\alpha)R^2}{r_\alpha}\frac{\omega(\Lambda^{\chi}s^{1+\chi}+s)}{s^2(\Lambda^{\chi}s^{1+\chi}+s+\nu r_\alpha^2)+\omega^2(\Lambda^{\chi}s^{1+\chi}+s+\nu r_\alpha^2)}, \tag{21}$$

$$\bar{u}_{2H}(r_\beta, s) = \frac{U\, J_1(Rr_\beta)\,\omega\,R}{r_\beta(s^2+\omega^2)} - \frac{UJ_1(Rr_\beta)R}{r_\beta}\frac{\omega(s+\Lambda^{\chi}s^{1+\chi})}{s^2(\Lambda^{\chi}s^{1+\chi}+s+\nu r_\beta^2)+\omega^2(\Lambda^{\chi}s^{1+\chi}+s+\nu r_\beta^2)}, \tag{22}$$

applying inverse Hankel transform on Eqs. 21-22 and using Appendix A (A5) and (A6), we get suitable series expansion as

$$\bar{u}_1(r,s) = \frac{r\Omega\omega}{s^2+\omega^2} - \frac{2\,\Omega\,\omega}{s^2+\omega^2}\sum_{\alpha=1}^{\infty}\frac{J_1(rr_\alpha)}{r_\alpha J_2(Rr_\alpha)}\sum_{s_1=0}^{\infty}(-\nu r_\alpha^2)^{s_1}\sum_{s_2=0}^{\infty}\frac{(-\Lambda^{\chi})^{s_2}\Gamma(s_1+s_2)}{s_2!\,\Gamma(s_1)s^{s_1-s_2\chi}}, \tag{23}$$

$$\bar{u}_2(r,s) = \frac{U\omega}{s^2+\omega^2} - \frac{2\,U\,\omega}{R(s^2+\omega^2)}\sum_{\beta=1}^{\infty}\frac{J_0(rr_\beta)}{r_\beta J_1(Rr_\beta)}\sum_{s_1=0}^{\infty}(-\nu r_\alpha^2)^{s_1}\sum_{s_2=0}^{\infty}\frac{(-\Lambda^{\chi})^{s_2}\Gamma(s_1+s_2)}{s_2!\,\Gamma(s_1)s^{s_1-s_2\chi}}, \tag{24}$$

inverting Eqs. 23-24 by means of Laplace transform and using theorem of convolution product, we have final form of velocities in terms of Fox-H function as

$$u_1(r,t) = r\Omega H(t)\sin(\omega t) - 2\Omega H(t)\sum_{\alpha=1}^{\infty}\frac{J_1(rr_\alpha)}{r_\alpha J_2(Rr_\alpha)}\sum_{s_1=0}^{\infty}(-\nu r_\alpha^2)^{s_1}\int_0^t \sin\omega(t-\delta)\times \mathrm{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda^{\chi}}{t^{\chi}}\right)^{s_2}\middle|\begin{matrix}(1-s_1,1)\\(0,1),(1-s_1,0),(1-s_1,-\chi)\end{matrix}\right]d\delta, \tag{25}$$

$$u_2(r,t) = UH(t)\sin(\omega t) - \frac{2UH(t)}{R}\sum_{\beta=1}^{\infty}\frac{J_0(rr_\beta)}{r_\beta J_1(Rr_\beta)}\sum_{s_1=0}^{\infty}(-\nu r_\beta^2)^{s_1}\int_0^t \sin\omega(t-\delta)\times H_{1,3}^{1,1}\left[\left(-\frac{\Lambda^{\chi}}{t^{\chi}}\right)^{s_2}\middle|\begin{matrix}(1-s_1,1)\\(0,1),(1-s_1,0),(1-s_1,-\chi)\end{matrix}\right]d\delta \tag{26}$$

where, $H_{j,k+1}^{1,j}(Z)$ is generalized Fox H-function defined as (Abro et al., 2017b; Abro et al., 2017c)

$$t^n\sum_p^{\infty}\frac{(-\Phi)^p\prod_{k=1}^{m}\Gamma(i_k+I_kp)}{p!\prod_{k=1}^{n}\Gamma(j_k+J_kp)} = \boldsymbol{H}_{m,n+1}^{1,m}\left[\Phi\middle|\begin{matrix}(1-i_1,I_1),(1-i_2,I_2),\dots,(1-i_m,I_M)\\(0,1),(1-j_1,J_1),(1-j_2,J_2),\dots,(1-j_n,J_N)\end{matrix}\right]. \tag{27}$$

3.2. Shear stress

Applying Laplace transform on Eqs. 14-15 and keeping in mind Eqs. 10-11, we arrive at

$$\bar{\tau}_1(r,s) = \mu(1+\Lambda^{\chi}s^{\chi})^{-1}\left(\frac{\partial\bar{u}_1(r,s)}{\partial r} - \frac{\bar{u}_1(r,s)}{r}\right), \tag{28}$$

$$\bar{\tau}_2(r,s) = \mu(1+\Lambda^{\chi}s^{\chi})^{-1}\frac{\partial\bar{u}_2(r,s)}{\partial r}, \tag{29}$$

substituting Appendix A (A7) and (A8) and using facts of Bessel's function $J_0(rr_\alpha) = -r_\alpha J_1(rr_\alpha)$ also $rr_\beta J_1'(rr_\beta) - J_1(rr_\beta) = -rr_\beta J_2(rr_\beta)$ in Eqs. 28-29, we obtain simplified form as

$$\bar{\tau}_1(r,s) = 2\mu\Omega\omega \sum_{\alpha=1}^{\infty} \frac{J_2(rr_\alpha)}{J_2(Rr_\alpha)} \sum_{s_1=0}^{\infty} (-\nu r_\alpha^2)^{s_1} \sum_{s_3=0}^{\infty} (-\Lambda^\chi)^{s_3} \sum_{s_2=0}^{\infty} \frac{(-\Lambda^\chi)^{s_2}\Gamma(s_1+s_2)\omega}{s_2!\Gamma(s_1)(s^2+\omega^2)s^{s_1-s_2\chi-s_3\chi}}, \quad (30)$$

$$\bar{\tau}_2(r,s) = \frac{2\mu U\omega}{R} \sum_{\beta=1}^{\infty} \frac{J_1(rr_\beta)}{J_1(Rr_\beta)} \sum_{s_1=0}^{\infty} \left(-\nu r_\beta^2\right)^{s_1} \sum_{s_3=0}^{\infty} (-\Lambda^\chi)^{s_3} \sum_{s_2=0}^{\infty} \frac{(-\Lambda^\chi)^{s_2}\Gamma(s_1+s_2)\omega}{s_2!\Gamma(s_1)(s^2+\omega^2)s^{s_1-s_2\chi-s_3\chi}}, \quad (31)$$

inverting Eqs. 30-31 by means of Laplace transform and using theorem of convolution product, we have final form of shear stresses in terms of Fox-H function as

$$\tau_1(r,t) = 2\Omega H(t)\omega\mu \sum_{\alpha=1}^{\infty} \frac{J_2(rr_\alpha)}{J_2(Rr_\alpha)} \sum_{s_1=0}^{\infty} (-\nu r_\alpha^2)^{s_1} \sum_{s_3=0}^{\infty} (-\Lambda^\chi)^{s_3} \int_0^t \sin\omega(t-\delta) \times \mathbf{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda^\chi}{t^\chi}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-\chi)}^{(1-s_1,1)}\right] d\delta \quad (32)$$

$$\tau_2(r,t) = \frac{2UH(t)\omega\mu}{R} \sum_{\beta=1}^{\infty} \frac{J_1(rr_\beta)}{J_1(Rr_\beta)} \sum_{s_1=0}^{\infty} (-\nu r_\alpha^2)^{s_1} \sum_{s_3=0}^{\infty} (-\Lambda^\chi)^{s_3} \int_0^t \sin\omega(t-\delta) \times \mathbf{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda^\chi}{t^\chi}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-\chi)}^{(1-s_1,1)}\right] d\delta. \quad (33)$$

- **Case-II: For cosine oscillations:** Implementing identical algorithm, we also investigated the analytical solutions of cosine oscillations as

$$u_1(r,t) = r\Omega H(t)\cos(\omega t) - 2\Omega H(t) \sum_{\alpha=1}^{\infty} \frac{J_1(rr_\alpha)}{r_\alpha J_2(Rr_\alpha)} \sum_{s_1=0}^{\infty} (-\nu r_\alpha^2)^{s_1} \int_0^t \cos\omega(t-\delta) \times \mathbf{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda^\chi}{t^\chi}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-\chi)}^{(1-s_1,1)}\right] d\delta. \quad (34)$$

$$u_2(r,t) = UH(t)\cos(\omega t) - \frac{2UH(t)}{R} \sum_{\beta=1}^{\infty} \frac{J_0(rr_\beta)}{r_\beta J_1(Rr_\beta)} \sum_{s_1=0}^{\infty} \left(-\nu r_\beta^2\right)^{s_1} \int_0^t \cos\omega(t-\delta) \times \boldsymbol{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda^\chi}{t^\chi}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-\chi)}^{(1-s_1,1)}\right] d\delta. \quad (35)$$

$$\tau_1(r,t) = 2\Omega H(t)\omega\mu \sum_{\alpha=1}^{\infty} \frac{J_2(rr_\alpha)}{J_2(Rr_\alpha)} \sum_{s_1=0}^{\infty} (-\nu r_\alpha^2)^{s_1} \sum_{s_3=0}^{\infty} (-\Lambda^\chi)^{s_3} \int_0^t \sin\omega(t-\delta) \times \mathbf{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda^\chi}{t^\chi}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-\chi)}^{(1-s_1,1)}\right] d\delta, \quad (36)$$

$$\tau_2(r,t) = \frac{2UH(t)\omega\mu}{R} \sum_{\beta=1}^{\infty} \frac{J_1(rr_\beta)}{J_1(Rr_\beta)} \sum_{s_1=0}^{\infty} (-\nu r_\alpha^2)^{s_1} \sum_{s_3=0}^{\infty} (-\Lambda^\chi)^{s_3} \int_0^t \sin\omega(t-\delta) \times \mathbf{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda^\chi}{t^\chi}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-\chi)}^{(1-s_1,1)}\right] d\delta. \quad (37)$$

4. Special solutions

4.1. Solutions of Maxwell fluid for ordinary differential operator

Case-I: For sine oscillations: In order to retrieve the solutions for sine and cosine oscillations, we substitute $\chi \rightarrow 1$ in Eqs. 25-26, 34-35 and Eqs. 32-33, 36-37, we arrive

$$u_1(r,t) = r\Omega H(t)\sin(\omega t) - 2\Omega H(t) \sum_{\alpha=1}^{\infty} \frac{J_1(rr_\alpha)}{r_\alpha J_2(Rr_\alpha)} \sum_{s_1=0}^{\infty} (-\nu r_\alpha^2)^{s_1} \int_0^t \sin\omega(t-\delta) \times \mathbf{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda}{t}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-1)}^{(1-s_1,1)}\right] d\delta, \quad (38)$$

$$u_2(r,t) = UH(t)\sin(\omega t) - \frac{2UH(t)}{R} \sum_{\beta=1}^{\infty} \frac{J_0(rr_\beta)}{r_\beta J_1(Rr_\beta)} \sum_{s_1=0}^{\infty} \left(-\nu r_\beta^2\right)^{s_1} \int_0^t \sin\omega(t-\delta) \times \boldsymbol{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda}{t}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-1)}^{(1-s_1,1)}\right] d\delta. \quad (39)$$

$$\tau_1(r,t) = 2\Omega H(t)\omega\mu \sum_{\alpha=1}^{\infty} \frac{J_2(rr_\alpha)}{J_2(Rr_\alpha)} \sum_{s_1=0}^{\infty} (-\nu r_\alpha^2)^{s_1} \sum_{s_3=0}^{\infty} (-\Lambda)^{s_3} \int_0^t \sin\omega(t-\delta) \times \mathbf{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda}{t}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-1)}^{(1-s_1,1)}\right] d\delta, \quad (40)$$

$$\tau_2(r,t) = \frac{2UH(t)\omega\mu}{R} \sum_{\beta=1}^{\infty} \frac{J_1(rr_\beta)}{J_1(Rr_\beta)} \sum_{s_1=0}^{\infty} (-\nu r_\alpha^2)^{s_1} \sum_{s_3=0}^{\infty} (-\Lambda)^{s_3} \int_0^t \sin\omega(t-\delta) \times \mathbf{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda}{t}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-1)}^{(1-s_1,1)}\right] d\delta. \quad (41)$$

- **Case-II: For cosine oscillations**

$$u_1(r,t) = r\Omega H(t)\cos(\omega t) - 2\Omega H(t) \sum_{\alpha=1}^{\infty} \frac{J_1(rr_\alpha)}{r_\alpha J_2(Rr_\alpha)} \sum_{s_1=0}^{\infty} (-\nu r_\alpha^2)^{s_1} \int_0^t \cos\omega(t-\delta) \times \mathbf{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda}{t}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-1)}^{(1-s_1,1)}\right] d\delta, \quad (42)$$

$$u_2(r,t) = UH(t)\cos(\omega t) - \frac{2UH(t)}{R} \sum_{\beta=1}^{\infty} \frac{J_0(rr_\beta)}{r_\beta J_1(Rr_\beta)} \sum_{s_1=0}^{\infty} \left(-\nu r_\beta^2\right)^{s_1} \int_0^t \cos\omega(t-\delta) \times \boldsymbol{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda}{t}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-1)}^{(1-s_1,1)}\right] d\delta. \quad (43)$$

$$\tau_1(r,t) = 2\Omega H(t)\omega\mu \sum_{\alpha=1}^{\infty} \frac{J_2(rr_\alpha)}{J_2(Rr_\alpha)} \sum_{s_1=0}^{\infty} (-\nu r_\alpha^2)^{s_1} \sum_{s_3=0}^{\infty} (-\Lambda)^{s_3} \int_0^t \sin\omega(t-\delta) \times \mathbf{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda}{t}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-1)}^{(1-s_1,1)}\right] d\delta, \quad (44)$$

$$\tau_2(r,t) = \frac{2UH(t)\omega\mu}{R} \sum_{\beta=1}^{\infty} \frac{J_1(rr_\beta)}{J_1(Rr_\beta)} \sum_{s_1=0}^{\infty} (-\nu r_\alpha^2)^{s_1} \sum_{s_3=0}^{\infty} (-\Lambda)^{s_3} \int_0^t \sin\omega(t-\delta) \times \mathbf{H}_{1,3}^{1,1}\left[\left(-\frac{\Lambda}{t}\right)^{s_2}\Big|_{(0,1),(1-s_1,0),(1-s_1,-1)}^{(1-s_1,1)}\right] d\delta. \quad (45)$$

4.2. Solutions of Newtonian fluid

- **Case-I: For sine oscillations:** In order to retrieve the solutions for sine and cosine oscillations, we substitute $\Lambda \rightarrow 0$ in Eqs. 25-26, 34-35 and Eqs. 32-33, 36-37, we obtained Newtonian solutions

$$u_1(r,t) = r\Omega H(t)\sin(\omega t) - 2\Omega\omega H(t) \sum_{\alpha=1}^{\infty} \frac{J_1(rr_\alpha)}{r_\alpha J_2(Rr_\alpha)} \int_0^t \sin\omega(t-\delta)\, Exp(-\nu r_\alpha^2)t\, d\delta, \quad (46)$$

$$u_2(r,t) = UH(t)\sin(\omega t) - \frac{2U\omega H(t)}{R} \sum_{\beta=1}^{\infty} \frac{J_0(rr_\beta)}{r_\beta J_1(Rr_\beta)} \int_0^t \sin\omega(t-\delta)\, Exp(-\nu r_\alpha^2)t\, d\delta, \quad (47)$$

$$\tau_1(r,t) = 2\mu\Omega\omega H(t) \sum_{\alpha=1}^{\infty} \frac{J_2(rr_\alpha)}{J_2(Rr_\alpha)} \int_0^t \sin\omega(t-\delta)\, Exp(-\nu r_\alpha^2)t\, d\delta, \quad (48)$$

$$\tau_2(r,s) = \frac{2\mu U\omega H(t)}{R} \sum_{\beta=1}^{\infty} \frac{J_1(rr_\beta)}{J_1(Rr_\beta)} \int_0^t \sin\omega(t-\delta)\, Exp(-\nu r_\alpha^2)t\, d\delta \quad (49)$$

- **Case-II: For cosine oscillations**

$$u_1(r,t) = r\Omega H(t)\cos(\omega t) - 2\Omega H(t) \sum_{\alpha=1}^{\infty} \frac{J_1(rr_\alpha)}{r_\alpha J_2(Rr_\alpha)} \int_0^t \cos\omega(t-\delta)\, Exp(-\nu r_\alpha^2)t\, d\delta \quad (50)$$

$$u_2(r,t) = UH(t)\cos(\omega t) - \frac{2UH(t)}{R}\sum_{\beta=1}^{\infty}\frac{J_0(rr_\beta)}{r_\beta J_1(Rr_\beta)}\int_0^t \cos\omega(t-\delta)\,Exp(-\nu r_\alpha^2)t\,d\delta, \tag{51}$$

$$\tau_1(r,t) = 2\mu\Omega H(t)\sum_{\alpha=1}^{\infty}\frac{J_2(rr_\alpha)}{J_2(Rr_\alpha)}\int_0^t \cos\omega(t-\delta)\,Exp(-\nu r_\alpha^2)t\,d\delta, \tag{52}$$

$$\tau_2(r,s) = \frac{2\mu UH(t)}{R}\sum_{\beta=1}^{\infty}\frac{J_1(rr_\beta)}{J_1(Rr_\beta)}\int_0^t \cos\omega(t-\delta)\,Exp(-\nu r_\alpha^2)t\,d\delta. \tag{53}$$

It is also worth pointed out that our general solutions can also be retrieved for ordinary Newtonian fluid when relaxation time ($\Lambda^\chi = 0$) is zero and fractional parameter ($\chi = 1$) is assumed to be equal to one Fetecau et al. (2008). In continuation, the solution obtained by Fetecau et al. (2008) can be retrieved by substituting $\omega = 0$ in Eqs. 46-49 or Eqs. 50-53.

5. Numerical results and discussions

In this portion, our purpose is to analyze few rheological parameters for helical flow of fractionalized viscoelastic fluid in helically moved cylinder numerically. The cylinder starts to oscillate and rotate about its axis with angular as well as oscillating velocities corresponding with shear stresses.

The exact solutions are investigated for both velocities and shear stresses along with imposed conditions $u_1(r,0) = u_2(r,0) = \frac{\partial u_1(r,0)}{\partial t} = \frac{\partial u_2(r,0)}{\partial t} = 0$ and $u_1(R,t) = R\Omega H(t)\,sin(\omega t)$ or $cos(\omega t)$, and $u_2(R,t) = UH(t)sin\omega t/cos\omega t$. The general solutions are presented in terms Fox **H**-function $\boldsymbol{H}_{j,k+1}^{1,j}(Z)$ with few particular cases, namely fractionalized second grade, ordinary second grade, ordinary Maxwell fluid and Newtonian fluid. Under these circumstances, the rheology is considered with distinct parameters and material limitations.

In order to have an insight of physical interpretation for helically moved cylinder, the graphical analysis is depicted for knowing the hidden differences and similarities on fluid flow. However, the major outcomes are enumerated below:

- Fig. 2 is prepared for the influences of time parameter on angular velocity $u_1(r,t)$ and oscillating velocity $u_2(r,t)$. Both velocities have qualitatively identical behavior for increase in time t. It is found that angular velocity $u_1(r,t)$ has sequestrating behavior and oscillating velocity $u_2(r,t)$ has smattering behavior on the whole domain of cylinder surface.

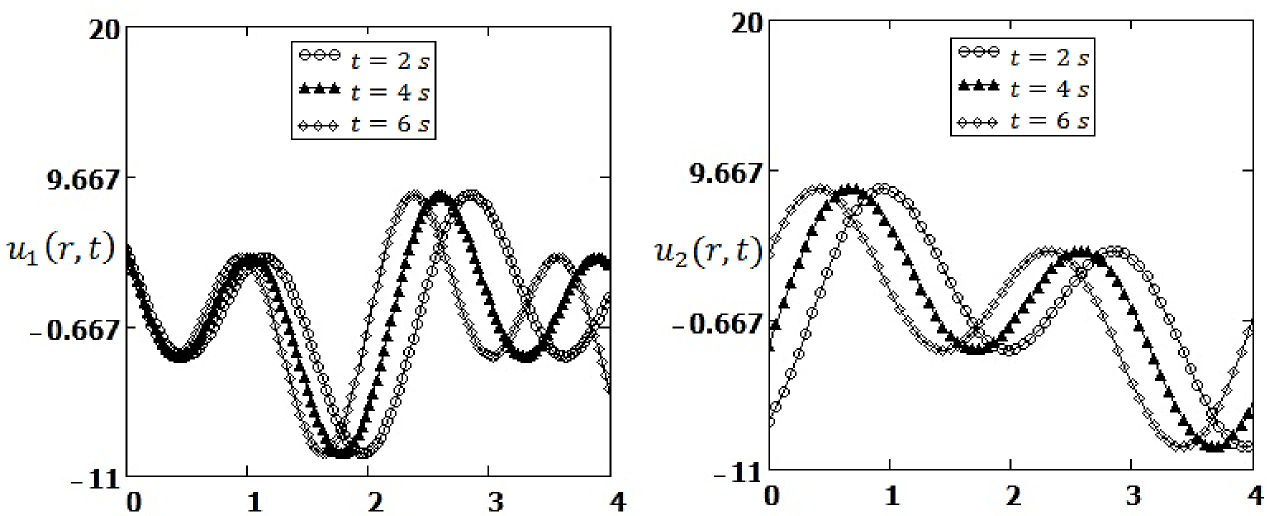

Fig. 2: Plot of angular and oscillating velocities for t

- Fig. 3 is depicted to show the effects of viscosity parameter ν on angular velocity $u_1(r,t)$ and oscillating velocity $u_2(r,t)$. It is noticed that that as viscosity increase; both velocities have oscillating behavior of fluid flow in scattering manners. It is also pointed out that oscillating velocity $u_2(r,t)$ has dominant behavior of flow in comparison with angular velocity $u_1(r,t)$.
- Fig. 4 is plotted for amplitude ω of angular velocity $u_1(r,t)$ and oscillating velocity $u_2(r,t)$, as expected the periodic response of fluid flow over the boundary of circular cylinder is observed. It is also clear that both velocities have distinct fluctuations; this may be due to the fact of imposed boundary conditions.
- The variation of radius of circular cylinder is displayed in Fig. 5 with range 0.1, 0.2, 0.3. It is noted that both velocities have oscillating behavior within insignificant interval. Also, it is observed that an angular velocity has shorter oscillations in comparison with oscillating velocity.

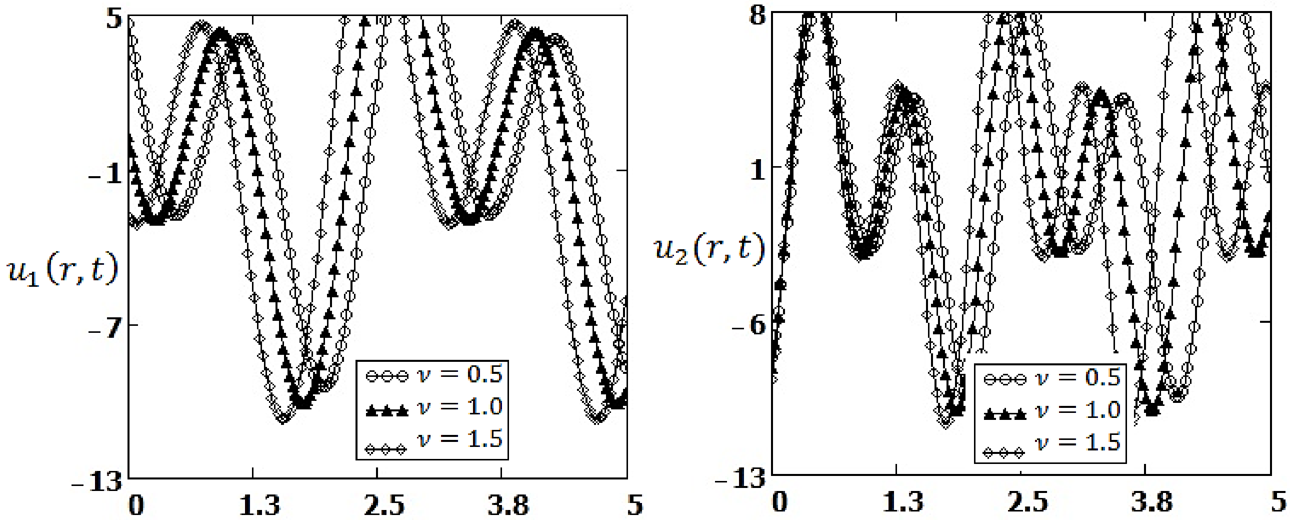

Fig. 3: Plot of angular and oscillating velocities for ν

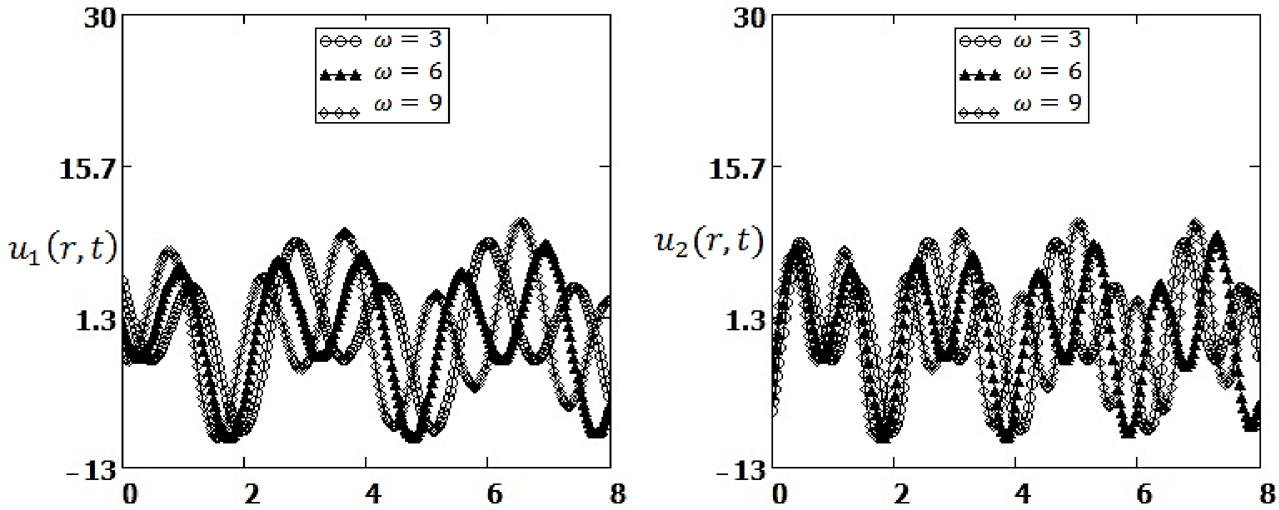

Fig. 4: Plot of angular and oscillating velocities for ω

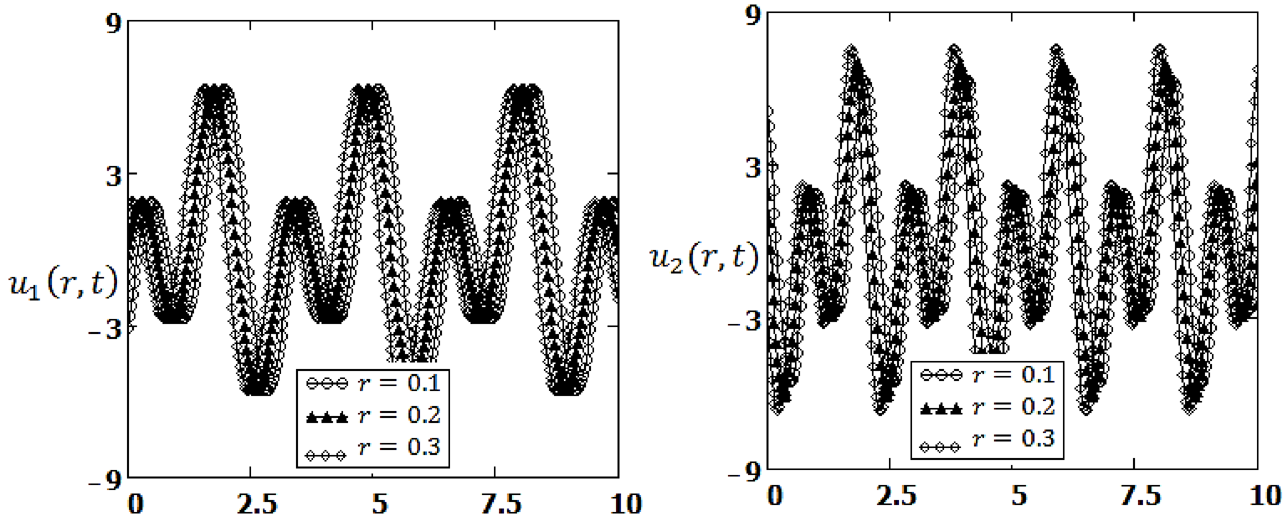

Fig. 5: Plot of angular and oscillating velocities for r

- Fig. 6 explains the hidden phenomenon of fractional parameter χ on both angular and oscillating velocities. Here, the behavior of both velocities is quite identical to each other. It is pointed out that oscillating velocity $u_2(r,t)$ moves rapidly in comparison with angular velocity $u_1(r,t)$. This may be due to fact of fractional order derivatives which examines the complete description of the memory effectively.
- Figs. 7 and 8 are drawn for the comparison of three ordinary as well as fractional models i-e (i) fractional and ordinary Maxwell fluid, (ii) fractional and ordinary second grade fluid and (iii) fractional and ordinary Newtonian fluid. In both figures,

angular velocity $u_1(r,t)$ and oscillating velocity $u_2(r,t)$ has opposite trend for fluid flow. It is observed that ordinary Newtonian fluid moves faster in angular velocity $u_1(r,t)$ and ordinary Maxwell moves faster in oscillating velocity. On the other hand, fractional Newtonian fluid moves faster in angular velocity $u_1(r,t)$ and fractional Maxwell moves faster in oscillating velocity. In brevity, a kingpin point in this comparison is the reciprocal behavior of fluid flows is observed either in ordinary models or in fractional models. The same phenomenon can be analyzed for shear stresses as well.

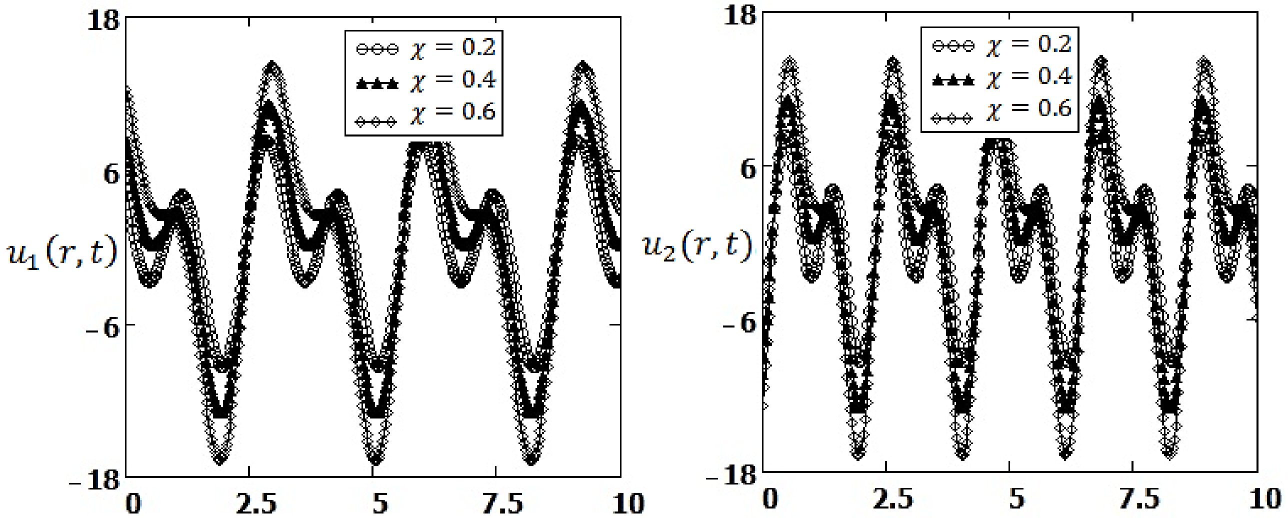

Fig. 6: Plot of angular and oscillating velocities for χ

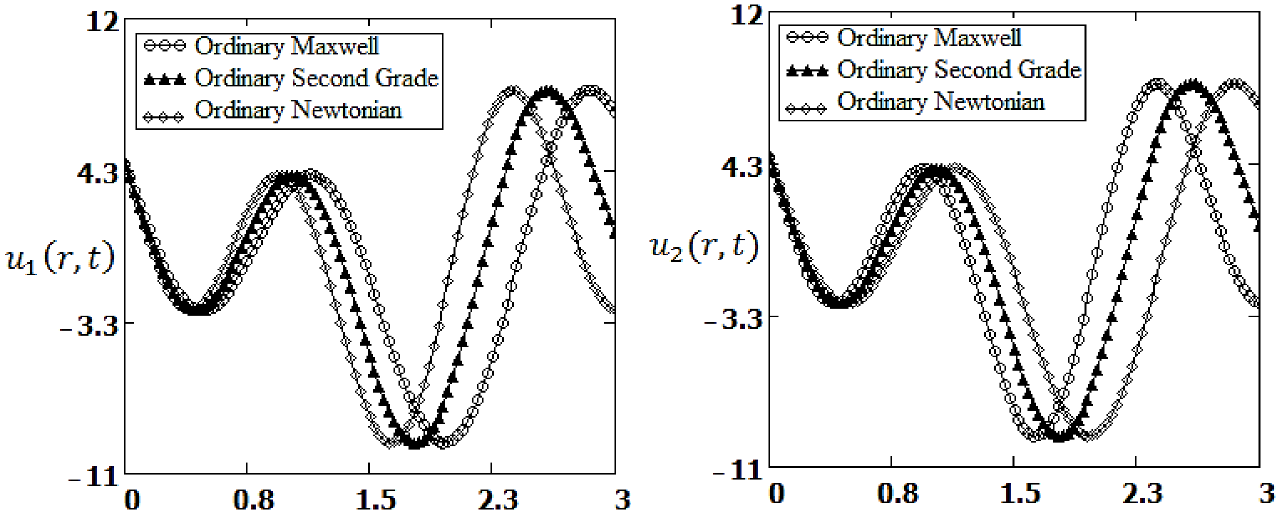

Fig. 7: Comparison of angular and oscillating velocities for three ordinary models

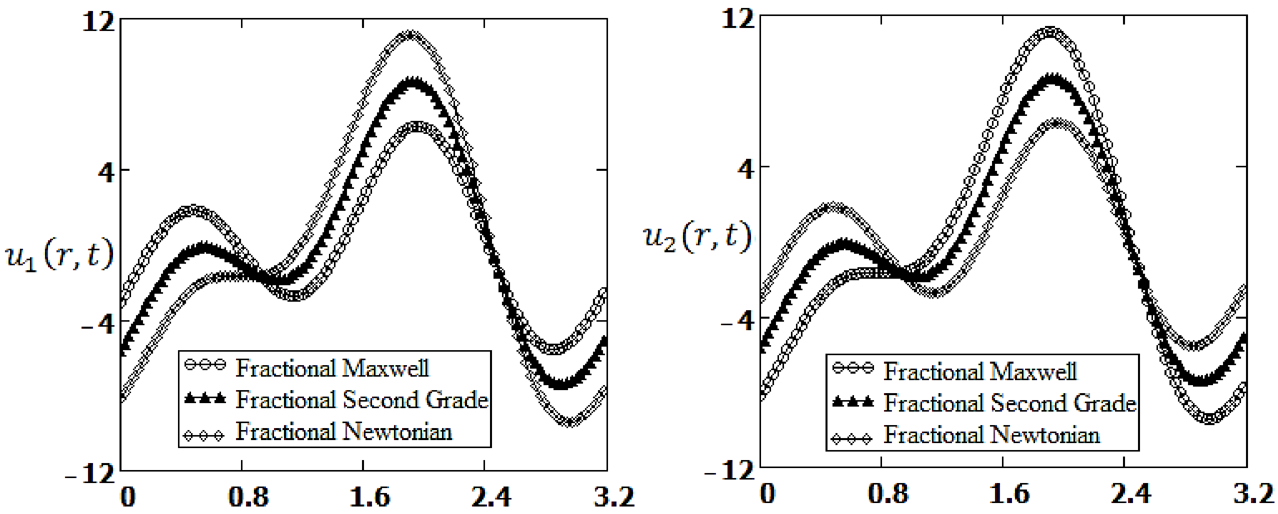

Fig. 8: Comparison of angular and oscillating velocities for three fractional models

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List of symbols

$u_1(r,t)$	Angular velocity $(m.s^{-1})$	$u_2(r,t)$	Oscillating velocity $(m.s^{-1})$
$\tau_1(r,t)$ or $\tau_2(r,t)$	Shear stress	$H^{1,j}_{j,k+1}(Z)$	Fox-H function
D^{χ}_{t}	Caputo fractional derivative	$H(t)$	Heaviside function
T	Cauchy stress tensor	S	Extra stress tensor
A	Rivilvin Ericksen tensor	L	Velocity gradient
r,θ,z	Cylindrical coordinates (m)	e_θ, e_z	Unit vectors
$-pI$	Spherical stress	μ	Viscosity of the fluid $(Pa.s)$
r_α, r_β	Hankel transform parameters	ω	Frequency
s	Laplace transform parameter	U, Ω	Non zero constants
t	Time parameter	ρ	Density of the fluid $(kg.m^{-3})$
χ	Fractional parameter	ν	Kinematic viscosity $(m^2.s^{-1})$
Λ	Relaxation time parameter	r	Radius of cylinder (m)
j,k	Indexes of Fox-H function	T	Transpose
$\Gamma(\blacksquare)$	Gamma function	s_1, s_2, s_3	Letting parameters
δ	Convolution parameter		

Appendix A. Finite Hankel transform

$$\int_0^R \left(\frac{\partial^2 \overline{u_1}}{\partial r^2} r J_1(rr_\alpha) + \frac{\partial \overline{u_1}}{\partial r} J_1(rr_\alpha) + \overline{u_1} J_1(rr_\alpha)\right) dr = (Rr_\alpha J_2(Rr_\alpha) - r_\alpha^2)\overline{u_{1H}}(r_\alpha, t), \tag{A1}$$

$$\int_0^R \left(\frac{\partial^2 \overline{u_2}}{\partial r^2} r J_0(rr_\beta) + \frac{\partial \overline{u_2}}{\partial r} J_0(rr_\beta)\right) dr = \left(Rr_\beta J_1(Rr_\beta) - r_\beta^2\right)\overline{u_{2H}}(r_\beta, t) \tag{A2}$$

$$\bar{u}_{1H}(r_\alpha, s) = \int_0^R \bar{u}_{1H}(r_\alpha, s)\, r\, J_1(rr_\alpha) dr, \tag{A3}$$

$$\bar{u}_{2H}(r_\beta, s) = \int_0^R \bar{u}_{2H}(r_\beta, s)\, r\, J_0(rr_\beta) dr, \tag{A4}$$

$$\bar{u}_1(r,s) = \frac{2}{R^2}\sum_{\alpha=1}^{\infty} \bar{u}_{1H}(r_\alpha, s)\frac{J_1(Rr_\alpha)}{J_2^2(Rr_\alpha)}, \quad \bar{u}_2(r,s) = \frac{2}{R^2}\sum_{\beta=1}^{\infty} \bar{u}_{2H}(r_\beta, s)\frac{J_0(Rr_\beta)}{J_1^2(Rr_\beta)}, \tag{A5}$$

$$\frac{R^2 J_2(Rr_\alpha)}{r_\alpha} = \int_0^R J_1(rr_\alpha) r^2\, dr \quad \text{and} \quad \frac{RJ_1(Rr_\beta)}{r_\beta} = \int_0^R J_0(rr_\beta) r\, dr, \tag{A6}$$

$$\frac{\partial \bar{u}_1(r,s)}{\partial r} - \frac{1}{r}\bar{u}_1(r,s) = 2\Omega\omega \sum_{\alpha=1}^{\infty} \frac{J_2(rr_\alpha)}{J_2(Rr_\alpha)} \frac{(s+\Lambda^\chi s^{\chi+1})}{s^2\left(\Lambda^\chi s^{1+\chi}+s+\nu r_\beta^2\right)+\omega^2\left(\Lambda^\chi s^{1+\chi}+s+\nu r_\beta^2\right)}, \tag{A7}$$

$$\frac{\partial \bar{v}_1(r,s)}{\partial r} = \frac{2U\omega}{R}\sum_{\beta=1}^{\infty} \frac{J_1(rr_\beta)}{J_1(Rr_\beta)} \frac{(s+\Lambda^\chi s^{\chi+1})}{s^2\left(\Lambda^\chi s^{1+\chi}+s+\nu r_\beta^2\right)+\omega^2\left(\Lambda^\chi s^{1+\chi}+s+\nu r_\beta^2\right)} \tag{A8}$$

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